Mutual Gravitational and Collisional Scattering of Bodies in the Asteroid Belt

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(Z. Naturforsch. **23 a**, 791—795 [1968]; received 30 June 1967)

A calculation is made of the cumulative effect of numerous small orbital perturbations caused by gravitational and collisional deflections of meteorites by other bodies in the asteroid belt. It is found that in a time equal to the age of the solar system the mean change in perihelion distance caused by gravitational perturbation is of the order of 10^{-4} A. U.

This phenomenon is consequently of negligible importance in removing meteorites from the asteroid belt. This conclusion is in disagreement with a result reported in the recent literature. The calculated mean change in perihelion distance caused by multiple collisional scattering is somewhat higher, about 10^{-3} A. U. It is conceivable that a small fraction of very strong objects such as iron meteorites in highly inclined and eccentric orbits may be deflected into Mars-crossing orbits by this mechanism.

A serious difficulty in the asteroidal theory of the origin of meteorites is that of removing the asteroidal fragments from the asteroid belt and placing them into earth-crossing orbits. This has been discussed by a number of authors, e. g. Arnold [1], Anders [2], Öpik [3], and Wetherill and Williams [4]. One conceivable mechanism whereby this might be accomplished is by multiple scattering: either by repeated gravitational perturbations by the numerous small asteroids passing in its vicinity, or by repeated "jostling" by mechanical collisions with even smaller bodies. It has been reported [5] that the first of these mechanism suffices to remove meteorite-sized bodies from the asteroid belt on a time scale of 10^5 years. In the present paper it will be shown that the first of these mechanism is ineffective in placing meteorites into earth-crossing orbits. The second is probably also unimportant, but may be significant for very strong objects in highly inclined or eccentric orbits.

In order for a body to be placed into an earth-crossing orbit it is necessary that its perihelion distance q be changed from a typical asteroidal value of $1.8-3.0$ A.U. to less than 1.0 A.U. The relationship between this necessary change in perihelion (δq) and the changes in the velocity components of the body will first be found.

Since $q=a(1-e)$, ($a=$ semi-major axis, $e=$ eccentricity) the change in q associated with small changes δa and δe will be

$$\delta q = \delta a(1-e) - a\,\delta e \tag{1}$$

The heliocentric components of velocity of the body will be given by

$$U_x{}^2 = \gamma^2\left(2 - \frac{1}{A} - A(1-e^2)\right), \tag{2}$$

$$U_y{}^2 = \gamma^2 A(1-e^2)\cos^2 i\,, \tag{3}$$

$$U_z{}^2 = \gamma^2 A(1-e^2)\sin^2 i \tag{4}$$

and the total velocity will be given by

$$U^2 = U_x{}^2 + U_y{}^2 + U_z{}^2 = \gamma^2(2-1/A) \tag{5}$$

where the coordinate system is chosen so that the x direction is radial, the y direction is perpendicular to the x direction in the plane of the solar system (positive in the direction of direct planetary motion) and the z direction is perpendicular to the plane of the solar system. $A=a/\varrho$, where ϱ is the distance from the sun; $\gamma=\sqrt{GM/\varrho}$ where G is the gravitational constant and M the mass of the sun; and i is the inclination relative to the plane of the solar system.

By differentiation of (2) and (5) it is found that

$$\delta e = \frac{1}{e}\left[\frac{U\,\delta U}{\gamma^2}\left(2-\frac{2}{A}-\frac{U_x{}^2}{\gamma^2}\right)+\frac{U_x\,\delta U_x}{A\,\gamma^2}\right], \tag{6}$$

$$\delta a = 2\,U\,\delta U\,a^2/\gamma^2\,\varrho \tag{7}$$

and by substitution into (1)

$$\delta q = \frac{U\,\delta U}{\gamma^2}\left[\frac{2\,a^2(1-e)}{\varrho} - \frac{a}{e}\left(2-\frac{2}{A}-\frac{U_x{}^2}{\gamma^2}\right)\right] - \frac{U_x}{e}\frac{\delta U_x}{\gamma^2}\,\varrho\,. \tag{8}$$

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[2] E. Anders, Space Sci. Rev. **3**, 583 [1964].
[3] E. J. Öpik, Adv. Astron. Astrophys. **4**, 302 [1966].
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The quantities $U\,\delta U/\gamma^2$ and $U_x\,\delta U_x/\gamma^2$ are dimensionless, so δq will be given in A.U. if a and ϱ are expressed in this unit.

It is convenient to write (8) in the form

$$\delta q = \zeta_1 \frac{U\,\delta U}{\gamma^2} - \zeta_2 \frac{U_x\,\delta U_x}{\gamma^2} \tag{9}$$

where ζ_1 and ζ_2 are functions of a, e, and the position of the meteorite in its orbit, Θ.

The case of multiple gravitational scattering will be treated first. The new velocity components resulting from gravitational perturbation through an angle ψ in the reference frame of the perturbing body are given by:

$$U_{x2}' = U_x' \cos\psi - \frac{U'\,U_y'}{\sqrt{U_x'^2+U_y'^2}} \sin\psi\cos\eta - \frac{U_z'\,U_x'\sin\psi\sin\eta}{\sqrt{U_x'^2+U_y'^2}}, \tag{10}$$

$$U_{y2}' = U_y' \cos\psi + \frac{U'\,U_x'}{\sqrt{U_x'^2+U_y'^2}} \sin\psi\cos\eta - \frac{U_z'\,U_y'\sin\psi\sin\eta}{\sqrt{U_x'^2+U_y'^2}}, \tag{11}$$

$$U_{z2}' = U_z' \cos\psi + \sqrt{U_x'^2+U_y'^2}\,\sin\psi\sin\eta \tag{12}$$

where η is the azimuth of the intersection of the trajectory of the body under study with a target circle perpendicular to this trajectory, and having the perturbing body at its center. The primed velocities U_x', U_y', U_t', and U' refer to the velocities of the perturbed body in the rest frame of the perturbing body.

When the angle of deflection is small, $\sin\psi \sim \psi$ and $\cos\psi \sim 1$. Then

$$\delta U_x' \cong \frac{\psi}{\sqrt{U_x'^2+U_y'^2}} \left[-U'\,U_y'\cos\eta - U_z'\,U_x'\sin\eta\right], \tag{13}$$

$$\delta U_y' \cong \frac{\psi}{\sqrt{U_x'^2+U_y'^2}} \left[U'\,U_x\cos\eta - U_z'\,U_y'\sin\eta\right], \tag{14}$$

$$\delta U_z' \cong \psi\sqrt{U_x'^2+U_y'^2}\,\sin\eta\,. \tag{15}$$

The deflection angle ψ will be given by

$$\sin(\psi/2) = \frac{1}{1+b\,U'^2/G\,m} \tag{16}$$

where G is the gravitational constant, m is the mass of the perturbing body, and b is the distance of closest approach of the two bodies.

For small deflections, (16) is equivalent to

$$\psi \cong 2\,G\,m/b\,U'^2. \tag{17}$$

If the perturbing body is significantly larger than the perturbed body, the velocity of the larger body will not change very much. In this case

$$\delta U_x \cong \delta U_x',\ \ \delta U_y \cong \delta U_y',\ \ \delta U_z \cong \delta U_z'. \tag{18–20}$$

For smaller perturbing bodies the changes in the heliocentric velocities will be smaller than the primed quantities.

The change in the total velocity can be found by differentiating U^2 and substitution of (18), (19), and (20):

$$\delta U = \frac{U_x}{U}\delta U_x' + \frac{U_y}{U}\delta U_y' + \frac{U_z}{U}\delta U_z' \tag{21}$$

which can be introduced into (9) to obtain

$$\delta q = \frac{U_x}{\gamma^2}(\zeta_1-\zeta_2)\,\delta U_x' + \frac{U_y\,\zeta_1}{\gamma^2}\,\delta U_y' + \frac{U_z\,\zeta_1}{\gamma^2}\,\delta U_z'\,. \tag{22}$$

Because the azimuth η, appearing in (13), (14), and (15), can range from 0 to 2π, negative values of $\delta U_x'$, $\delta U_y'$, and $\delta U_z'$ are just as probable as positive values, and on the average δq will be zero. However, the mean squared change, $\overline{\delta q^2}$, will not average to zero, and it is thereby possible for meteorites to "diffuse" out of the asteroid belt by random walk.

Averaging over al values of the azimuth, the mean square change in perihelion distance will be

$$\overline{\delta q^2} = \frac{\overline{\delta U_x'^2}(\zeta_1-\zeta_2)^2\,U_x^2}{\gamma^4} + \frac{\overline{\delta U_y'^2}\,\zeta_1^2\,U_y^2}{\gamma^4} + \frac{\overline{\delta U_z'^2}\,\zeta_1^2\,U_z^2}{\gamma^4} \tag{23}$$

where the terms proportional to the cross products such as $\delta U_x'\,\delta U_y'$ average to zero when averaged over all values of the azimuth η.

By use of expressions (13), (14), (15), and (17)

$$\overline{\delta q^2} = \frac{2\,G^2\,m^2}{b^2}\left\{\frac{(\zeta_1-\zeta_2)^2\,U_x^2}{\gamma^4}\,\frac{\overline{(U'^2\,U_y'^2+U_z'^2\,U_x^2)}}{U'^4(U_x'^2+U_y'^2)} + \frac{\zeta_1^2\,U_y^2\,\overline{(U'^2\,U_x'^2+U_z'^2\,U_y'^2)}}{\gamma^4\,U'^4(U_x'^2+U_y'^2)} + \frac{\zeta_1^2\,U_z^2}{\gamma^4}\,\frac{\overline{U_x'^2+U_y'^2}}{U'^4}\right\} \tag{24}$$

for a perturbing body of mass m at a distance b.

The total squared changed in perihelion will be given by multiplying (24) by the number of bodies of mass m at distance b with given orbital elements and summing over the complete range of masses, impact parameters, and orbital elements.

It will be assumed that the asteroidal masses are distributed in accordance with a power law, which, when expressed in terms of their radii gives

$$dn = C\, r^{-p}\, dr \tag{25}$$

for the number of perturbing bodies per unit volume with radii between r and $r + dr$. In an earlier study [6] an expression was obtained for the probability per year (P_F) of a body with orbital elements (a, e, i) passing within a distance b of another body with orbital elements (a_0, e_0, i_0). It was found that this probability was proportional to b^2 and thus can be written in the form $P_F = P_{i0}\, b^2$. The probability per year of it passing at a distance between b and $b + db$ will be $2\, b\, P_{i0}\, db$. The squared change in perihelion, per year due to *all* bodies between mass m and mass $m + dm$ passing at distances between b and $b + db$ and having the same orbital elements will be

$$\overline{\delta^2 q_1} = \frac{4\, G^2}{b} \left(\tfrac{4}{3}\pi\right)^2 \sigma^2 r^6\, C\, r^{-p}\, P_{i0}\, \Phi\, db \tag{26}$$

where σ is the average density of the perturbing bodies, and Φ is the expresion in braces in (24).

The averaging of the product P_{i0} over all values of the orbital elements of the perturbing bodies for a given set of elements of the perturbed body could be performed numerically if desired. However in this paper we are concerned only with evaluating the importance of this scattering mechanism. For this purpose we will replace P_{i0} with the average quantity P_i, obtained by averaging numerically P_{i0} over a field of asteroids having the orbital elements of the 127 numbered of absolute magnitude 9.0 or brighter, assuming the orbital elements of the perturbed body are those of "Astrid", a hypothetical average asteroid with $a = 2.75$ A.U., $e = 0.2727$, and $i = 0.2760$ radians. Fig. 1 is a histogram showing the distribution of relative velocities between an object having the orbital elements of Astrid and a field of asteroids with the orbital elements of the 327 numbered asteroids brighter than absolute magnitude 10.0. In obtaining a numerical value for

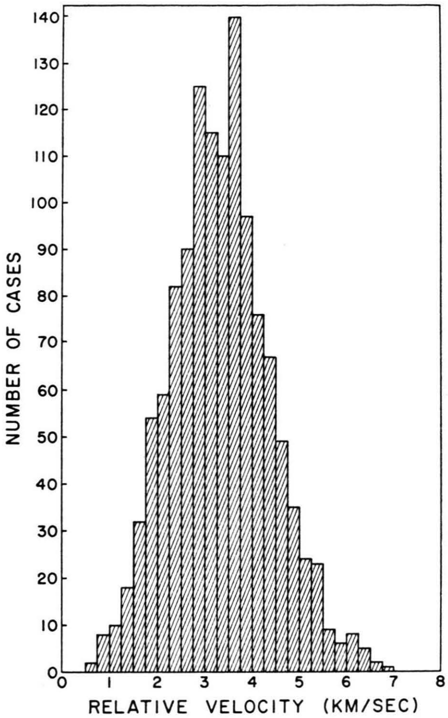

Fig. 1. Relative velocities of the hypothetical average asteroid Astrid (a=2.75 A.U., e=0.2727, i=0.2760 radians) with the 327 numbered asteroids brighter than absolute photographic magnitude 10.0 The velocities were computed at the point of intersection of the two orbits, assuming random relative longitudes of their nodes with respect to the invariable plane of the solar system.

the average value of Φ, the orbital elements of Astrid are again used for the perturbed body, and the average values $U = 3.5$ km/sec consisting of components $U_x = U_y = U_z = 2.02$ km/sec are used for the relative velocities. The final conclusion does not depend upon this assumption, and even unreasonable choices of these velocities will not significantly alter the conclusion.

Expression (26) may then be summed over the entire range of radii and impact parameters with the result:

$$\overline{\overline{\delta^2 q_1}} = \frac{64}{9}\, \frac{\pi^2\, G^2\, C\, P_i\, \sigma^2}{(7-p)}\, (r_2{}^{7-p} - r_1{}^{7-p})\; \overline{\Phi}\, \ln \frac{b_{\max}}{b_{\min}}\,. \tag{27}$$

Since r_1 will be of the order of the meteorite radius and r_2, the largest perturbing body will be ~ 100 km in radius, $r_2 \gg r_1$ and the contribution of $r_1{}^{7-p}$ in (27) is negligible for reasonable values of p (see previous discussion [6]). It will be assumed that the index $p = 3$, although the conclusion does

[6] G. Wetherill, J. Geophys. Res. **72**, 2429 [1967].

not depend upon this assumption. Then

$$\overline{\overline{\delta q_1{}^2}} = \frac{16\pi^2}{q} G^2 C P_i \sigma^2 r_2{}^4 \overline{\Phi} \ln \frac{b_{\max}}{b_{\min}}. \qquad (28)$$

Because of their occurrence in the logarithm, this expression will be insensitive to the choice of $b_{\max}$ and $b_{\min}$.

It is necessary to keep units consistent in evaluating (28). This may be done conveniently by writing $\overline{\Phi} = \overline{\Phi'}/U'^4$ where now Φ' consists of dimensionless quantities multiplied by function of ζ_1 and ζ_2, which, as previously noted, will give q in A.U. if a and ϱ are expressed in A.U. The remaining factor in (28) will then have the dimensions of yr^{-1}, essentially consisting of an average squared deflection angle per perturbing body per year multiplied by the number of perturbing bodies. If the quantities entering into this factor are expressed in any consistent set of units, q^2 will be given in A.U.^{2} yr^{-1}.

The value of C appropriate to the index $p = 3$ will be taken to be 2.81×10^{15} cm^2, $\Theta = \pi/2$, $P_i = 2.5 \times 10^{-28}$ cm^{-2} yr^{-1} (as discussed previously[6]), $\sigma = 3.5$ g/cm^3, $r_2 = 10^7$cm, $b_{\max} = 10^{10}$ cm, $b_{\min} = 10^2$ cm. The calculated value of $\overline{\Phi'}$ is 0.322. Substitution of these numerical quantities into (28) gives the result:

$$\overline{\overline{\delta q_1{}^2}} \cong 2.6 \times 10^{-18} \text{ A.U.}^2 \text{ yr}^{-1}. \qquad (29)$$

In the age of the solar system, i. e. 4.5×10^9 years, the total accumulated mean square change in perihelion will be

$$\overline{\overline{\delta q_T{}^2}} \cong 1.2 \times 10^{-8} \text{ A.U.}^2 \qquad (30)$$

and the root mean square change in perihelion will be:

$$(\overline{\overline{\delta q_T{}^2}})^{1/2} \cong 1.1 \times 10^{-4} \text{ A.U.} \qquad (31)$$

It may thus be seen that this mechanism will be ineffectual in removing meteorites from the asteroid belt even given times as long as the age of solar system. A corollary of this conclusion is that the approximation made by SITTE[5] (ignoring the fact that the meteorite is constrained by the gravitational attraction of the sun to move in a Keplerian orbit), is invalid.

The importance of the contribution of mutual mechanical collisions between asteroids can be evaluated more simply. The collisions which need to be considered are those between a meteorite sized body and smaller colliding bodies, which upon collision change the velocity of the meteorite. The largest colliding bdies which need be considered are those of radius $r \cong 0.5\,R$, where R is the radius of the meteorite, since larger colliding bodies may be expected to destroy the meteorite by extensive fragmentation. For an order of magnitude calculation it may be assumed that the momentum change of the meteorite is equal to the momentum of the colliding body. Therefore the change in velocity δU of the meteorite will be

$$\delta U \sim \frac{r^3}{R^3} U' \qquad (32)$$

where U' is the relative velocity of the two bodies. From (9)

$$\delta q^2 = \frac{U_x{}^2(\zeta_1 - \zeta_2)(\delta U_x)^2}{\gamma^4} + \frac{U_z{}^2 \zeta_1{}^2 (\delta U_z{}^2)}{\gamma^4} + \frac{U_y{}^2 \zeta_1{}^2 (\delta U_y{}^2)}{\gamma^4}. \qquad (33)$$

The components of δU can be written in the form

$$\delta U_x = \frac{r^3}{R} U' \cos \xi_x, \qquad (34)$$

$$\delta U_y = \frac{r^3}{R} U' \cos \xi_y, \qquad (35)$$

$$\delta U_z = \frac{r^3}{R} U' \cos \xi_z. \qquad (36)$$

The total increment in $\delta U_y{}^2$ per year due to collisions with smaller bodies of radius between r and $r + \mathrm{d}r$ will be

$$\delta U_{y1}{}^2 = \frac{P_i r^6}{R^4} C r^{-p} \overline{U'^2 \cos^2 \xi_y}, \qquad (37)$$

which when integrated over all permissible values of the radius of the colliding body gives

$$\overline{\delta^2 U_{y1}} = \frac{P_i C}{R^4} \frac{\overline{U'^2 \cos^2 \xi_y}}{(7-p)} ((0.5\,R)^{7-p} - r_{\min}{}^{7-p}). \qquad (38)$$

Again taking $p = 3$, and noting that $r_{\min} \sim 10^{-4}$ cm $\ll R$,

$$\overline{\delta^2 U_{y1}} = \frac{U_y{}^2}{64} P_i C \overline{U'^2 \cos^2 \xi_y}. \qquad (39)$$

The contribution of the second term to δq^2 in (33) will be

$$\overline{\delta q_{1y}{}^2} = \frac{U_y{}^2 \zeta_1{}^2}{64\,\gamma^4} P_i C \overline{\cos^2 \xi_y U'^2}. \qquad (40)$$

Since $U_y{}^2 > U_x{}^2 \cong U_z{}^2$, this term will predominate.

By substitution of the numbers used for the case of gravitational scattering it is found that the root mean square change in perihelion distance in 4.5 $\times 10^9$ years is

$$(\overline{\delta q_1^2})^{1/2} \cong 2 \times 10^{-3} \text{ A.U.} \quad (41)$$

Therefore, for collisions between objects in typical meteorite orbits, the effect of this second mechanism is also seen to be small. For stone meteorites, it is probably smaller still, because they are unlikely to survive collision with projectiles as large as half their radius, as was assumed in these calculations. However by assuming higher values of the index p in the power law for the size distribution as well as highly inclined and eccentric orbits, which lead to higher relative velocities U' it may be possible to raise $(\overline{\delta q_1^2})^{1/2}$ by one or two orders of magnitude. Even stronger bodies would be required to survive this increased number of these higher velocity impacts, but it may be that meteoritic iron possesses this necessary strength. In this case multiple collisional scattering could provide a small supply of iron meteorites in Mars-crossing orbits in addition to those present at the time of origin of the solar system.

Acknowledgements

I wish to thank GENNVIEVE KURTIN for typing of the manuscript. This work was supported by NASA Grant NsG-313-63.

Oscillographic Polarography in Molten Nitrates

I. The Behaviour of the Ions Tl^+, Cd^{++}, Pb^{++}, Zn^{++} and Ni^{++} in the Eutectic $LiNO_3$–$NaNO_3$–KNO_3

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(Z. Naturforsch. **23 a**, 795—799 [1968]; received 20 November 1967)

The polarographic investigation of Pb^{++}, Cd^{++}, Tl^+, Zn^{++} and Ni^{++} ions in a KNO_3-$LiNO_3$-$NaNO_3$ molten eutectic mixture was carried out by conventional and oscillographic (single-sweep) polarography. Experiments were carried out in the temperature range 150—200 °C, with a dropping mercury electrode as the cathode and a Ag/AgCl electrode as anode and reference. For ions with reversible behaviour (Cd^{++}, Pb^{++}, Tl^+) the polarographic diffusion coefficients calculated from conventional polarography data by means of the Ilkovič equation agree, within experimental errors, with those obtained from oscillographic data by the Randles-Sevčik equation at low potential change rates. At higher potential change rates deviations of the experimental peak currents from their theoretical values were observed. This is associated with a marked distortion of the wave and indicates a certain degree of kinetic control. The temperature dependence of the diffusion coefficients in the investigated range is the same for the ions with reversible behaviour and the corresponding activation energy is $E = -8.4$ kcal/mole.

1. Introduction

The feasibility of using the dropping mercury electrode (d. m. e.) in polarographic studies of molten nitrate melts was demonstrated some years ago by STEINBERG and NACHTRIEB [1]; their work indicated that in such low melting systems the d. m. e. could be used without particular difficulties and that the Ilkovič equation was quite valid.

The purpose of this work is to test, on the same system, the validity of the equation first established by RANDLES [2] and SEVČIK [3] for the case of oscillographic polarography at linear voltage scanning. More recently MATSUDA and AYABE [4] carried out a complete analytical treatment of the single sweep oscillographic polarography including the cases of reversible, irreversible and "quasi" reversible electrode reactions; in this work the expressions derived by MATSUDA are used.

For the case of the reversible electroreduction of a cation MATSUDA obtained the following equations:

$$(i_p)_r = 0{,}447\, n\, F\, A\, C\, D^{\frac{1}{2}} (n\, F\, v/R\, T)^{\frac{1}{2}}, \quad (1)$$

$$(E_p)_r = E\tfrac{1}{2} - 1{,}11\, R\, T/n\, F, \quad (2)$$

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